A Practical Review of Linear Ordinary Differential Equations

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Overview

In your Dynamic Systems & Control course, you are going to need to solve elementary differential equations like a pro. You will need to solve the equations and interpret the solutions with very little effort. I know that all of you have taken a differential equations course in which all the necessary background has been covered. However, it has been my experience that students entering MEE 322 are a bit rusty.

The purpose of these notes is to refresh your memory. I have intentionally left out a bunch of the theory that would normally accompany a thorough discussion of differential equations. You can refer back to your differential equations text for that. Here, I try to cover just the most important aspects that you will need to be fluent with if you are going to succeed in your Dynamic Systems & Control course.

These notes have gone through several revisions. They have gotten better primarily through student feedback. I hope that you send me an email (`bcoller@niu.edu`) if you run across any mistakes. Also, if something is not clear, you can help me clarify the discussion. I would be grateful for any constructive criticism that you can provide.

Chapter 1

What Does it Mean to Solve a Differential Equation?

Differential equations come in many different flavors. If you skim through the chapter titles of this handbook, you'll see descriptors such as *homogeneous, nonhomogeneous, first order, second order, constant coefficient, linear, and nonlinear* to characterize different types. In this handbook, we're going to discuss several different techniques to "solve" many different flavors of differential equations. When learning (or relearning) these different solution techniques, it's easy to lose track of what it means to have a solution. Therefore, we begin this handbook with such a discussion.

1.1 An Example

Suppose we have a differential equation of the form

$$\frac{d}{dt}y(t) - 2y(t) = 0, \qquad \text{or} \qquad \frac{d}{dt}y(t) = 2y(t). \tag{1.1}$$

In words, this equation simply states that the derivative of the function $y(t)$ with respect to t is equal to the function itself, multiplied by two. In this set of notes we will usually think of the independent variable t as *time*. So we might think of $y(t)$ as a *position* or *length* that changes with respect to time, and the derivative dy/dt as a *speed*.

A *solution* to the differential equation is simply a function $y(t)$ that satisfies the equality in (1.1). Suppose, for example, we take the function $y(t) = \sin(t)$. Then, the derivative is $dy/dt = \cos(t)$. Substituting these into (1.1), we get

$$\cos(t) \stackrel{?}{=} 2\sin(t).$$

Surely, these two terms are equal to each other for *some* values of t. However, for $y(t) = \sin(t)$ to be a solution to the differential equation, the equality must be true for *all* values of t. Therefore, $y(t) = \sin(t)$ is NOT a solution to (1.1).

Now let's try a different function: $y(t) = e^{2t}$. Notice that the derivative is $dy/dt = 2\,e^{2t}$. Substituting these into (1.1), we find

$$2\,e^{2t} \stackrel{?}{=} 2(e^{2t}).$$

Aha!! The equality above is satisfied for *all* values of t. Therefore, $y(t) = e^{2t}$ *is* a solution.

1.2 An Example with Multiple (independent) Solutions

Let's look an another example:

$$\frac{d^2}{dt^2}y(t) + 2y(t) = 0, \qquad \text{or} \qquad \frac{d^2}{dt^2}y(t) = -2y(t). \tag{1.2}$$

To solve this differential equation is to find a function $y(t)$ whose *second derivative* is equal to the function itself, multiplied by -2.

What if we try $y(t) = e^{2t}$ again? Notice that $d^2y/dt = 4\,e^{2t}$. Substituting this into (1.2), we obtain

$$4\,e^{2t} \stackrel{?}{=} -2(e^{2t}).$$

Thus, it does **not** satisfy the differential equation.

If we try $y(t) = e^{\sqrt{2}\,t}$, the substitution yields

$$2\,e^{\sqrt{2}\,t} \stackrel{?}{=} -2(e^{\sqrt{2}\,t}).$$

Because of the unbalanced minus sign above, this doesn't work either.

However, notice what happens if we try $y(t) = \sin\left(\sqrt{2}\,t\right)$. The second derivative is $d^2y/dt^2 = -2\,\sin(\sqrt{2}\,t)$. Subtitution into (1.2) gives

$$-2\,\sin(\sqrt{2}\,t) \stackrel{?}{=} -2\left(\sin(\sqrt{2}\,t)\right).$$

Ooooooh! Here's the solution.

Hummmm. In the sentence above, I should not have said that $\sin(\sqrt{2}\,t)$ is ***the*** solution. It gives the impression that it is the *only* solution. You can check for yourself that $y(t) = \cos(\sqrt{2}\,t)$ is also a solution. Therefore, the differential equation (1.2) has two completely different (independent) solutions.

1.3 Take-Aways

Below is a list of what I feel to be the most important points that I want you to "take away" from Chapter 1.

1. If $y(t)$ is a solution to a differential equation, you should be able to substitute it into the differential equation and the equality should hold for all values of the independent variable. (In our examples, the independent variable was t.)

2. A differential equation may have multiple, independent solutions.

Chapter 2

First Order, Constant Coefficient, Linear, Homogeneous Differential Equations

One way to find a solution to a differential equation is to simply keep trying different functions until you find one that works. Depending on how good a guesser you are, this could take a long time. Fortunately, for certain types of differential equations, there is a systematic approach for finding a solution... all the solutions.

2.1 Finding a Solution

We'll start off by revisiting the first differential equation we encountered in the previous chapter:

$$\frac{d}{dt}\,y(t) - 2\,y(t) = 0. \tag{2.1}$$

As we'll discuss later, differential equations similar to (2.1) tend to have solutions that are exponential functions. Therefore, we'll try a solution of the form

$$y(t) = e^{st}, \tag{2.2}$$

where s is some constant that we have to determine. In your previous differential equations course, you may have called this constant r or λ. In Dynamic Systems & Control, though, we always call it s.

So let's try it out. The derivative of (2.2) is $dy/dt = se^{st}$. Substituting this and (2.2) into (2.1), we get

$$se^{st} - 2e^{st} \stackrel{?}{=} 0.$$

Factoring out the exponential on the left side, we get

$$(s-2)e^{st} \stackrel{?}{=} 0.$$

Regardless of what s and t are, the term e^{st} is always a positive number. (Please check this for yourself.) Therefore, the only way that the equality can be satisfied is if

$$s - 2 = 0. \tag{2.3}$$

Equation (2.3) is so important that it has a name: the *characteristic equation.* It is the equation that allows us to determine the constant s. In this simple example, the characteristic equation tells us that s must be 2. Upon substituting $s = 2$ into (2.2), a solution to the differential equation (2.1) is given by

$$y(t) = e^{2t}. \tag{2.4}$$

Of course, this is the same solution we found in Section 1.1.

2.1.1 Another Simple Example

Now let's try another simple example:

$$\frac{d}{dt}\,y(t) + y(t) = 0. \tag{2.5}$$

Upon substitution of $y(t) = e^{st}$ into the new differential equation, we get the following characteristic equation:

$$s + 1 = 0. \tag{2.6}$$

Did you get this for the characteristic equation? Do you understand exactly how the get the characteristic equation? If not, go back and re-read the appropriate parts in the previous section.

It is clear that $s = -1$ solves the characteristic equation and hence a solution to the differential equation (2.5) is

$$y(t) = e^{-t}. \tag{2.7}$$

Couldn't be simpler, eh?

2.2 Some Nomenclature

The title of this chapter describes these differential equations as being "first order," "constant coefficient," "linear," and "homogeneous." Wow, that's a mouth full.

We say that Equations (2.1, 2.5) are ***first order*** because the highest order derivatives are *first* derivatives with respect to time.

We use the term ***constant coefficient*** because the coefficients in front of the $y(t)$ and dy/dt terms in (2.1, 2.5) are constant. An example of a non-constant coefficient differential is given by

$$6\,\frac{d}{dt}y(t) + 2t^2\,y(t) = 0 \tag{2.8}$$

Because the coefficient $2t^2$ varies with time, this is *not* a constant coefficient differential equation.

The differential equations we have seen so far are all ***linear*** because the dependent variable $y(t)$ and its derivative appear linearly. Notice that there are no terms that look like $y^2(t)$ or $\sin(y(t))$. Notice that although there is a quadratic t^2 term in (2.8), the differential equation is still linear because $y(t)$ and its derivative appear linearly. Linear equations are going to be important for us in Dynamic Systems and Control.

Finally, we say that the differential equations discussed in this chapter are ***homogeneous*** because all the terms in the equations have a $y(t)$ or derivative of $y(t)$. An example of a differential equation which is *not* homogeneous is

$$7\,\frac{d}{dt}y(t) + 2\,y(t) = t^3.$$

The t^3 term makes it non homogeneous.

Whenever we have first order, constant coefficient, linear, homogeneous differential equations, like those we discussed in Sections 2.1 and 2.1.1, the solution will take the form $y(t) = e^{st}$. Upon substituting this form of a solution (with undetermined s) into the differential equation, one obtains a characteristic polynomial similar to (2.3, 2.6). By finding the root of the characteristic polynomial, you can determine the characteristic exponent s.

You may check for yourself that a solution of the form $y(t) = e^{st}$ will *not* work for Equation (2.8).

2.3 What Do the Solutions Look Like?

Earlier in this chapter, we obtained two solutions to two differential equations. So far it's all about mathematics. In Dynamic Systems & Control, the differential equations are going to represent physical processes. Solutions to the differential equations are going to represent how physical quantities are changing in time. (Remember, we're interpreting our independent variable t as time.) The solution $y(t)$ might "grow" or "decay" or "oscillate". It might not change at all with time. To represent how things change in time, you will find your graphing/plotting skills to be very valuable in Dynamic Systems & Control.

Let's begin with the two solutions we found earlier in the chapter: $y_a(t) = e^{2t}$ and $y_b(t) = e^{-t}$? What do the graphs of these two functions look like?

Remember from your grade-school days that e is just a constant, a special constant. Its value is $e = 2.7182818284590451...$ So at time "zero," we have $y_a(t = 0) = e^{2 \cdot 0} = e^0 = 1$. Furthermore, $y_a(1) = e^2 = 7.3891$, and $y_a(2) = e^4 = 54.5982$. Table 2.1 shows more values of $y_a(t) = e^{2t}$ as well as values of the other solution $y_b(t) = e^{-t}$.

Table 2.1: Solutions $y(t)$ evaluated at five different values of time, t. I am only presenting this for illustrative purposes. Normally, I wouldn't expect (or even want) you to generate a table like this.

t	$y_a(t) = e^{2t}$	$y_b(t) = e^{-t}$
0.0	$e^0 = 1.0000$	$e^{-0} = 1/e^0 = 1.0000$
1.0	$e^2 = 7.3891$	$e^{-1} = 1/e^1 = 0.3679$
2.0	$e^4 = 54.598$	$e^{-2} = 1/e^2 = 0.1353$
3.0	$e^6 = 403.43$	$e^{-3} = 1/e^3 = 0.0497$
4.0	$e^8 = 2981.0$	$e^{-4} = 1/e^4 = 0.0183$

If we plot the solutions graphically, we can see the trends of the solutions more clearly. (See Figure 2.1.) The $y_a(t) = e^{2t}$ solution quickly increases. After just four units of time (perhaps 4 seconds), the solution grows almost 3000 times larger than it began. And it continues to grow. We say the that the solution "blows up exponentially fast".

In contrast, the $y_b(t) = e^{-t}$ solution "decays exponentially fast." After just four units of time, the solution is less than 2% of its original value. As time increases, the solutions continues to shrink toward zero.

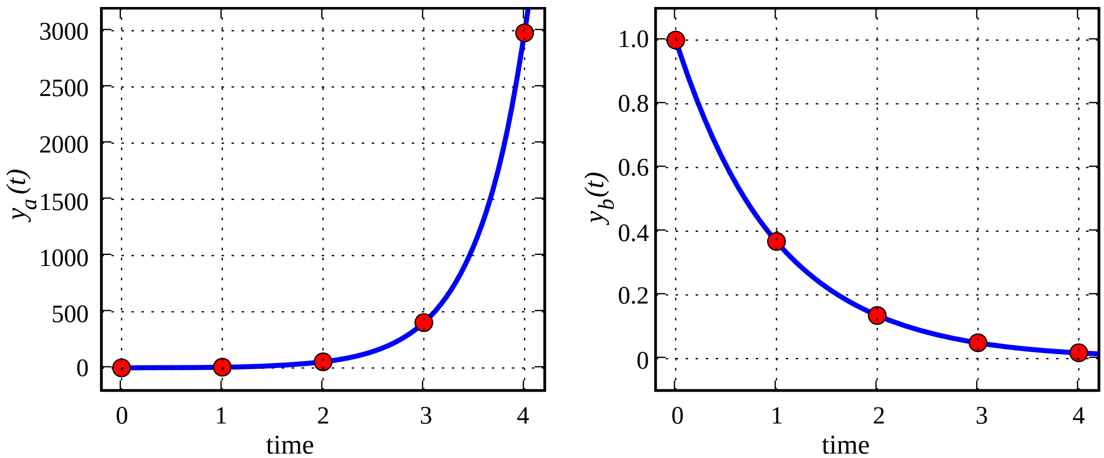

Figure 2.1: Plots of the two solutions $y_a(t)$ and $y_b(t)$, including the data points from Table 2.1.

Notice that there are no oscillations in either of these solutions. We see monotonic[1] growth in one case and monotonic decay in the other.

2.4 Rates of Growth and Decay

To examine growth and decay of differential equations more thoroughly, let's consider the following four differential equations:

$$\frac{dy}{dt} + 2y = 0, \qquad 2\frac{dy}{dt} + y = 0, \qquad 2\frac{dy}{dt} - 3y = 0, \quad \text{and} \quad \frac{dy}{dt} - 3y = 0. \tag{2.9}$$

As discussed in Section 2.2, the method of solving these equations is the same. By assuming a solution of the form $y(t) = e^{st}$, we get the following four characteristic equations (in order) that must be satisfied:

$$s + 2 = 0, \qquad 2s + 1 = 0, \qquad 2s - 3 = 0, \qquad \text{and} \qquad s - 3 = 0.$$

Notice that you can generate the characteristic polynomial equation by simply "reading" it from the differential equation with very little thought at all. The roots to the characteristic polynomial can be calculated simply, also:

$$s = -2, \qquad s = -\frac{1}{2}, \qquad s = \frac{3}{2}, \qquad \text{and} \qquad s = 3.$$

The characteristic roots, then, appear in the exponents of the solutions to the differential equations (2.9):

$$y(t) = e^{-2t}, \qquad y(t) = e^{-t/2}, \qquad y(t) = e^{3t/2}, \qquad y(t) = e^{3t}.$$

Each of these solutions is plotted in Figure 2.2 (blue and purple curves). Also shown in Figure 2.2 are the solutions $y_b(t)$ and $y_a(t)$ (black dashed curves) from the previous section.

[1] "Monotonic" is a fancy word. A *monotonically* increasing function never decreases. A *monotonically* decreasing function never increases.

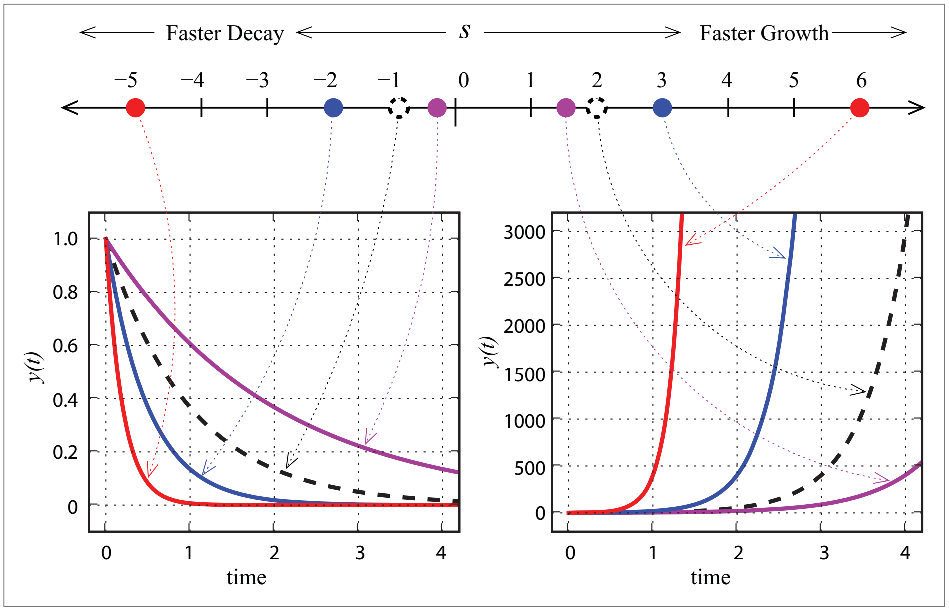

Figure 2.2: Solutions corresponding to a variety of different characteristic roots, s, illustrating how the roots affect the rates of decay and growth.

Two other curves (red) corresponding to characteristic roots $s = -5$ and $s = +6$ are shown in the figure as well.

Figure 2.2 is one that you should *understand thoroughly.* It depicts important solution features which are critical to control engineering. The following are questions you should be able to answer without hesitation. Answers are provided in Section 2.6.

1. Which (if any) of the solutions have oscillation?
2. Under what conditions (which types of characteristic roots) does the solution decay?
3. What types of characteristic roots correspond to solutions that decay more rapidly? Decay less rapidly?
4. Under what conditions (what types of characteristic roots) does the solution blow up, i.e. grow exponentially?
5. What types of characteristic roots correspond to solutions that grow more rapidly? Grow less rapidly?

Not shown in Figure 2.2 is what happens when s is zero. Following the solution procedure outlined earlier, a characteristic root of zero corresponds to a solution of the form:

$$y(t) = e^{0\,t} \equiv 1.$$

Therefore the solution is simply a constant. It does not change in time. It neither grows nor decays.

2.5 A Whole Family of Solutions

Back in Section 2.1.1, you may recall that we studied the differential equation

$$\frac{d}{dt}y(t) + y(t) = 0, \tag{2.5}$$

and found the characteristic root to be $s = -1$. Therefore, we found a solution to the differential equation as

$$y_b(t) = e^{-t}.$$

However, this is not the only solution. Consider, for example, the function

$$y_{b_2}(t) = 2\,e^{-t}, \quad \text{whose time derivative is} \quad \frac{d}{dt}y_{b_2}(t) = -2\,e^{-t}.$$

So if we substitute $y(t) = y_{b_2}(t)$ into (2.5), we get

$$\frac{d}{dt}y_{b_2}(t) + y_{b_2}(t) = -2\,e^{-t} + 2\,e^{-t} \equiv 0.$$

Therefore, $y_{b_2}(t)$ is also a solution. You may check for yourself that $y_{b_3}(t) = 3\,e^{-t}$ and $y_{b_4}(t) = 4\,e^{-t}$ are solutions also. In fact, $y_{b_c}(t) = c\,e^{-t}$, for any constant c is a solution to the differential equation (2.5). We say that there is a whole "family[2]" of solutions. There are an infinite number of solutions, one for each value of c in $y_{b_c}(t)$ (even negative values of c, and $c = 0$).

In Figure 2.3, we show several of these solutions, including the original solution, $y_b(t) = e^{-t}$ (dashed). It is clear from the plot that some solutions are "bigger" than others. However, all have the same relative decay rate.

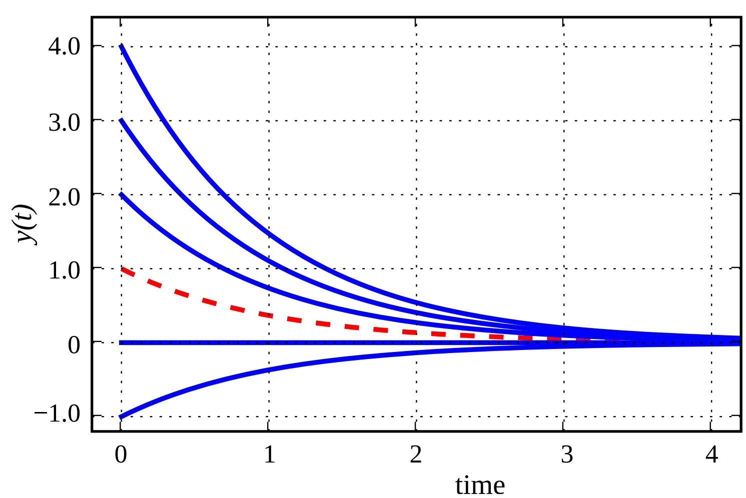

Figure 2.3: Multiple solutions to the differential equation (2.5). Our original solution $y_b(t) = e^{-t}$ is shown dashed.

Note: In that last paragraph, when I stated that all solutions in Figure 2.3 have the same "relative decay rate," did you know what that means? If not, did you try to figure it out? If

[2]The word "family" might sound corny, but it's the terminology that real mathematicians use.

you did know what it means, did you try to figure out why it is true? These are the types of questions you should ask yourself as you read these notes (or any notes, textbook, etc).

To show what I mean by "same relative decay rate," let's consider the general solution $y_{b_c}(t) = c\,e^{-t}$, where different values of the constant c produce the different curves in Figure 2.3. Let's evaluate this function at several times:

$$\begin{aligned}
y_{b_c}(t) &= c\,e^{-t}, \\
y_{b_c}(0) &= c\,e^{-0} = c \cdot 1.0 = c, \\
y_{b_c}(1) &= c\,e^{-1} = c \cdot 0.3679 = y_{b_c}(0) \cdot 0.3679, \\
y_{b_c}(2) &= c\,e^{-2} = \left(c\,e^{-1}\right) e^{-1} = y_{b_c}(1) \cdot 0.3679, \\
y_{b_c}(3) &= c\,e^{-3} = \left(c\,e^{-2}\right) e^{-1} = y_{b_c}(2) \cdot 0.3679, \\
y_{b_c}(4) &= c\,e^{-4} = \left(c\,e^{-3}\right) e^{-1} = y_{b_c}(3) \cdot 0.3679,
\end{aligned}$$

Therefore, for each unit of time, the solution y_{b_c} decreases by a factor of 0.3679. It does not matter what the value of c is. All solutions with characteristic root $s = -1$ lose 63.21% of their value for each unit of time[3].

2.5.1 Initial Conditions

Notice from Figure 2.3 that there is only one solution which passes through $y = 3$ at $t = 0$. Therefore, if we supplement the differential equation (2.5) with the initial condition $y(0) = 3$, then we get a single, "unique"[4] solution: $y(t) = y_{b_3}(t) = 3\,e^{-t}$. There is one "unique" solution for each initial condition.

2.6 Take-Aways

The take-away messages for Chapter 2 are outlined below.

1. To find the solution to a first order, constant coefficient, linear, homogeneous differential equation, seek a solution of the form $y(t) = e^{st}$. Upon substitution into the differential equation, you will find the characteristic polynomial, whose root (the characteristic root) gives the exponent s.

2. The characteristic polynomial for a first order differential equation has only one root. The root is a real number (i.e. not imaginary or complex).

3. Solutions to first order, constant coefficient, linear, homogeneous differential equations do not oscillate. They generally grow or decay exponentially. In one case, when the characteristic root is zero, the solution remains constant.

4. If the characteristic root is negative, then the solution decays exponentially. The more negative the root, the faster the decay rate.

5. If the characteristic root is positive, then the solution grows exponentially. The more positive the root, the faster the growth rate.

[3] 0.6321 = 1.0 - 0.3679

[4] Again, "unique" is a mathematical term. When we say a solution is unique, we mean that there is only one solution.

6. If the function $y(t) = e^{st}$, for some specific value of s, is a solution to a constant coefficient linear differential equation, then $y(t) = c\,e^{st}$ is also a solution, for *any* constant c. Thus, these differential equations have whole "families" of solutions. One can use the coefficient c to satisfy an initial condition.

Chapter 3

Second Order, Constant Coefficient, Linear, Homogeneous Differential Equations

In this chapter, we take our discussion of differential equations and their solutions just one step farther. In this one small step, the solutions become more rich.

3.1 An Example Problem

Let's start this chapter with an example:

$$\frac{d^2}{dt^2}y(t) + 3\,\frac{d}{dt}y(t) + 2\,y(t) = 0. \tag{3.1}$$

We say that this is a second order differential equation because the highest derivative is of order two.

To simplify our notation, we'll often write the differential equation above as

$$\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = 0, \qquad \text{or} \qquad \ddot{y} + 3\,\dot{y} + 2\,y = 0. \tag{3.2}$$

The "dots" over the dependent variable denote derivatives with respect to t (time). The number of dots denote the number of time derivatives. Also, sometimes we will get lazy and write $y(t)$ as simply y with the understanding that y depends on t.

Again, a "solution" to the differential equation is a function $y(t)$ which satisfies the equality in (3.1) for all time, t. As we did in Section 2.1, we shall assume that (3.1) has a solution of the form:

$$y(t) = c\,e^{st}, \tag{3.3}$$

where s is some constant that we have to determine. Note that in our assumed solution (3.3), we are also including the constant c. As discussed in Section 2.5, this is so that we can capture an entire family of solutions.

The first two derivatives of (3.3) are:

$$\dot{y}(t) = c\,s\,e^{st}, \qquad \text{and} \qquad \ddot{y}(t) = c\,s^2\,e^{st}.$$

Upon substituting these into (3.1), we get

$$\begin{aligned}\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) &= c\,s^2\,e^{st} + 3\,c\,s\,e^{st} + 2\,c\,e^{st} \\ &= c\,(s^2 + 3\,s + 2)\,e^{st} \\ &= 0.\end{aligned} \tag{3.4}$$

Notice that there are three potential ways to satisfy the equality above. We can set c to zero, set $(s^2 + 3\,s + 2)$ to zero, or set e^{st} to zero. However, recognizing that e^{st} is never zero, we can eliminate the last possibility immediately.

Note also, that if we set the constant c to zero, then the potential solution $y(t)$ is identically zero. While $y(t) = 0$ is a solution to (3.1), it is a rather boring solution. Mathematicians call it the "trivial solution."

Therefore, the only way we can make (3.3) work as a non-trivial solution is to choose an s which satisfies

$$s^2 + 3\,s + 2 = 0. \tag{3.5}$$

We call this the *characteristic equation* for (3.1). And since the polynomial $s^2 + 3\,s + 2$ conveniently factors into simple terms $(s+1)(s+2)$, we can easily find the *characteristic roots*:

$$s = -1, \qquad \text{and} \qquad s = -2.$$

Notice that by choosing $s = -1$ or $s = -2$, then the last equality of (3.4) is satisfied for all time t, regardless of the value of the constant c. Thus, upon substitution into (3.3), we see that we get two families of solutions:

$$y_1(t) = c_1\,e^{-t}, \qquad \text{and} \qquad y_2(t) = c_2\,e^{-2t}.$$

Here c_1 and c_2 are arbitrary constants. They can be anything.

It turns out that the sum of these two solutions,

$$\begin{aligned}y(t) &= y_1(t) + y_2(t) \\ &= c_1\,e^{-t} + c_2\,e^{-2t},\end{aligned} \tag{3.6}$$

is also a solution to (3.1). We can verify this easily by substituting (3.6) directly into (3.1):

$$\begin{aligned}\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) &= \frac{d^2}{dt^2}\Big(y_1(t) + y_2(t)\Big) + 3\,\frac{d}{dt}\Big(y_1(t) + y_2(t)\Big) + 2\,\Big(y_1(t) + y_2(t)\Big) \\ &= \Big(\ddot{y}_1(t) + \ddot{y}_2(t)\Big) + \Big(3\,\dot{y}_1(t) + 3\,\dot{y}_2(t)\Big) + \Big(2\,y_1(t) + 2\,y_2(t)\Big) \\ &= \Big(\ddot{y}_1(t) + 3\,\dot{y}_1(t) + 2\,y_1(t)\Big) + \Big(\ddot{y}_2(t) + 3\,\dot{y}_2(t) + 2\,y_2(t)\Big) \\ &= \Big(\quad 0 \quad\Big) + \Big(\quad 0 \quad\Big) \\ &= 0.\end{aligned}$$

The fourth equality above follows from the fact that $y_1(t)$ and $y_2(t)$ each satisfy the differential equation, individually. We think of Equation (3.6) as the *general solution.* As before, we can use the constants c_1 and c_2 to satisfy initial conditions. (See Section 3.3.)

3.1.1 A Glimpse at a Solution

In the previous section, we found that any *linear combination* of e^{-t} and e^{-2t} is a solution to the differential equation (3.1). In this section we will look at one specific solution. In (3.6), we'll set $c_1 = 3$ and $c_2 = -2$ to give

$$y(t) = 3\,e^{-t} - 2\,e^{-2t}. \tag{3.7}$$

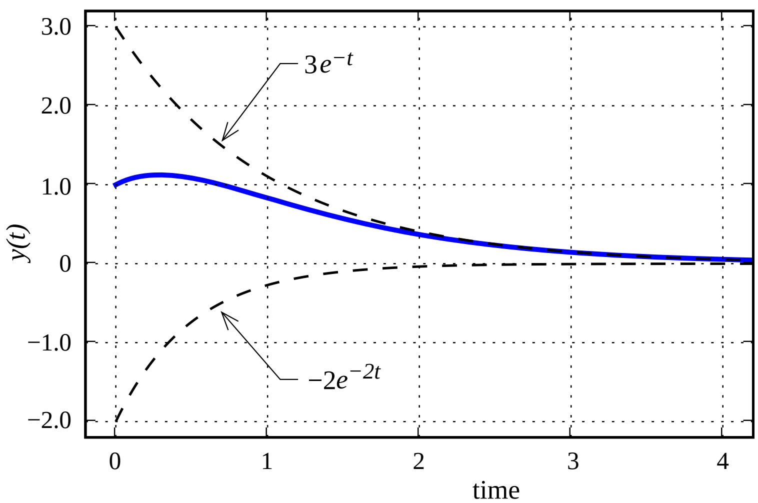

Figure 3.1: A solution (solid curve) to differential equation (3.1), along with its two parts (dashed).

A plot of this solution is provided in Figure 3.1 (solid curve). Also shown in the figure (dashed) are the two functions $3\,e^{-t}$ and $-2\,e^{-2t}$ so that you can see the role that they play. In particular, we make the following observations:

1. Observe that the two characteristic roots of differential equation (3.1) are both negative: $s = -1$ and $s = -2$. Therefore, the two parts of the solution ($c_1\,e^{-t}$ and $c_2\,e^{-2t}$) both decay to zero.

   The sum of two functions that decay to zero is itself a function that decays to zero. So the general solution $y(t)$ in (3.6) must decay to zero, regardless of the values of c_1 and c_2. In Figure 3.1, we see that our specific solution decays to zero.

2. Observe that even though the two individual parts of the solution($3\,e^{-t}$ and $-2\,e^{-2t}$) are monotonic, the sum of the two parts is *not necessarily* monotonic. This is clearly evident in the plot of Figure 3.1 which initially increases before decaying to zero.

   This happens because $3\,e^{-t}$ **decreases** toward zero, while the second term $-2\,e^{-2t}$ **increases** toward zero. And for a brief moment the second term dominates the rate. If we take a time derivative of $y(t)$ at $t = 0$ we get

   $$\dot{y}(0) = \left(-3\,e^{-t} + 4\,e^{-2t}\right)\Big|_{t=0} = -3 + 4 = 1$$

   Thus, the slope of the solution is initially positive. Notice the second term dominating the derivative for small t.

3. Finally, observe in Figure 3.1 that for large t, the solution approaches zero along the part $3\,e^{-t}$, and not the other part, $-2\,e^{-2t}$. Why? I'll let you think about it.

4. As you're thinking about the question above, you might also want to think about what would happen if the two characteristic roots were $s = 2$ and $s = 3$. What if the roots were $s = -2$, $s = 1$?

3.2 An Example with Complex Characteristic Roots

OK. Let's try another example. Consider the following differential equation:

$$\ddot{y}(t) + 4\,\dot{y}(t) + 104\,y(t) = 0. \tag{3.8}$$

If we seek a solution of the form $y(t) = c\,e^{st}$, then we get the following characteristic equation that must be satisfied:

$$s^2 + 4\,s + 104 = 0. \tag{3.9}$$

This polynomial is a bit more difficult to factor in one's head. So, let's rely on the trusty quadratic formula, where the roots of $a\,s^2 + b\,s + c = 0$ are given by

$$s_1,\ s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If you have forgotten the quadratic formula, it is time to recommit it to memory. Applying the formula to our characteristic equation (3.9), we obtain the roots

$$s = -2 + 10\,i, \qquad \text{and} \qquad s = -2 - 10\,i.$$

The roots are complex! Here, $i = \sqrt{-1}$ is the "imaginary" constant.[1]

Recall that the importance of the characteristic roots is that they tell us the exponents in the solution. Substituting the roots into our assumed solution $y(t) = c\,e^{st}$, we get the solutions

$$y_1(t) = c_1\,e^{(-2+i\,10)t}, \qquad \text{and} \qquad y_2(t) = c_2\,e^{(-2-i\,10)t}. \tag{3.10}$$

But what the heck is $e^{(-2+i\,10)t}$?

Let's manipulate it a bit:

$$e^{(-2+i\,10)t} = e^{-2t+i\,10t} = e^{-2t}\,e^{i\,10t}.$$

Now, you may recall something called Euler's formula that goes like

$$e^{i\,\beta} = \cos(\beta) + i\,\sin(\beta).$$

I'm not going to derive Euler's formula for you. You can look it up for yourself if you're curious[2]. We'll just put it into our complex exponential to find our solutions

$$y_1(t) = c_1\,e^{(-2+i\,10)t} = c_1\,e^{-2t}\big(\cos(10t) + i\,\sin(10t)\big) = c_1\big(e^{-2t}\cos(10t) + i\,e^{-2t}\sin(10t)\big)$$

and

$$y_2(t) = c_2\,e^{(-2-i\,10)t} = c_2\,e^{-2t}\big(\cos(10t) - i\,\sin(10t)\big) = c_2\big(e^{-2t}\cos(10t) - i\,e^{-2t}\sin(10t)\big).$$

These are a bit clunky, but these are solutions.

[1]A lot of control books denote the imaginary constant $\sqrt{-1}$ with the symbol j. I think this is because the field of control theory was once dominated by electrical engineers who reserved the symbol i for electric current. I have always used $i = \sqrt{-1}$, so I'm going to stick to it.

[2]The usual derivation uses Taylor series of the exponential, of sine, and of cosine.

3.2.1 What do we do with complex solutions?

So the general solution of Equation (3.8) is a complex function. While this is true mathematically, the result might not seem to have any connection to the real, physical world in which engineers work.

After all, Equation (3.8) is what we get when we derive the equation of motion for the simple mass-spring system shown in Figure 3.2, where the mass is 1 kg, viscous damping coefficient is 4 kg/s, and spring constant 104 N/m.

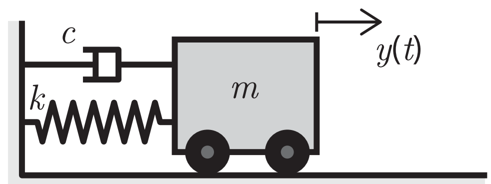

Figure 3.2: This simple mass-spring system yield equations of motion similar to (3.8).

In this case, $y(t)$ represents the displacement of the mass. So what does it mean to have a complex solution?:

$$y(t) = c_1\, e^{-2t} \left[\cos(10\, t) + i\, \sin(10\, t)\right] + c_2\, e^{-2t} \left[\cos(10\, t) - i\, \sin(10\, t)\right]. \tag{3.11}$$

Would it make sense to have a displacement of $y = (3 + 2\, i)$ meters? No, of course not.

So let's think about it more thoroughly. What if we were to collect the real and imaginary parts of (3.11)? We would end up with the following expression:

$$y(t) = (c_1 + c_2)\, e^{-2t}\, \cos(10\, t) + i\, (c_1 - c_2)\, e^{-2t}\, \sin(10\, t). \tag{3.12}$$

But note that c_1 and c_2 are just constants; they might even be complex constants. Therefore the $c_1 + c_2$ in (3.12) is just a constant. We'll call it c_3. Similarly $i\,(c_1 - c_2)$ is a different constant. We'll call it c_4. Therefore, we can write our solution (3.12) as

$$y(t) = c_3\, e^{-2t} \cos(10\, t) + c_4\, e^{-2t} \sin(10\, t). \tag{3.13}$$

Therefore, I claim that we can write the general solution to (3.8) by the real functions in (3.13) rather than the complex functions (3.11). When thinking of the differential equation 3.8 as representing the physics of a mass-spring-damper system shown in Figure 3.2, this *real* solution makes sense.

Verifying the real solution

Just in case you're skeptical (some students are) of the real solution we just derived, let's verify that it works. Recall from Chapter 1, we can simply substitute the function into the differential equation to see if it works.

Actually, we'll just check half the solution here. Define $y_3(t)$ as

$$y_3(t) = c_3\, e^{-2t} \cos(10\, t).$$

Its first and second derivatives are given by

$$\dot{y}_3(t) = c_3\big(-2\, e^{-2t} \cos(10\, t) - 10\, e^{-2t} \sin(10\, t)\big),$$

and

$$\begin{aligned}\ddot{y}_3(t) &= c_3\big(4\,e^{-2t}\cos(10\,t) + 20\,e^{-2t}\sin(10\,t) + 20\,e^{-2t}\sin(10\,t) - 100\,e^{-2t}\cos(10\,t)\big)\\ &= c_3\big(-96\,e^{-2t}\cos(10\,t) + 40\,e^{-2t}\sin(10\,t)\big).\end{aligned}$$

In both cases, we used the product rule for taking derivatives. Upon substitution of these into the left side of the differential equation (3.8), we get

$$\begin{aligned}\ddot{y}_3(t) + 4\,\dot{y}_3(t) + 104\,y_3(t) &= c_3\big(-96\,e^{-2t}\cos(10\,t) + 40\,e^{-2t}\sin(10\,t)\big) +\\ &\qquad 4\,c_3\big(-2\,e^{-2t}\cos(10\,t) - 10\,e^{-2t}\sin(10\,t)\big) +\\ &\qquad 104\,c_3\,e^{-2t}\cos(10\,t)\\ &= c_3\big(0\,e^{-2t}\cos(10\,t) + 0\,e^{-2t}\sin(10\,t)\big)\\ &= 0.\end{aligned}$$

Therefore, $y_3(t)$, all by itself, is a solution. I'll leave it to you to verify that $y_4(t) = c_4\,e^{-2t}\sin(10\,t)$, all by itself, is also a solution. Putting these two pieces together, we get the general solution (3.13).

3.2.2 Rules of Thumb for Complex Roots

We just spent two pages describing what happens when you get complex characteristic roots. Once you *understand* how it works, you do not need to go through all the steps outlined on the the previous two pages. Instead, you just need to follow a few simple rules:

1. *Complex roots come in conjugate pairs.* That is, whenever we get a complex root $s = \sigma + i\omega$ as a root to the characteristic equation, then the conjugate $s = \sigma - i\omega$ is also a root. This is a result of something called the Complex Conjugate Root Theorem.

2. *Solutions corresponding to complex characteristic roots.* Whenever we have a pair of complex characteristic roots of the form $s = \sigma \pm i\omega$, we get solutions of the form

$$\boxed{y(t) = c_1\,e^{\sigma t}\cos(\omega\,t) + c_2\,e^{\sigma t}\sin(\omega\,t).} \tag{3.14}$$

   Thus the real part, σ, of the root appears in the exponential as a growth rate or decay rate (depending on whether σ is positive or negative). The imaginary part, ω, of the characteristic root serves as a frequency of oscillation in the sine and cosine.

Please read this (twice). When you get a differential equation with complex roots, it is *not* necessary to go through the steps between Equations (3.10) and (3.12). After finding the complex characteristic roots, you may jump directly to (3.14) to write the general solution.

3.2.3 A Glimpse at an Oscillatory Solution

Before Section 3.2, all the characteristic roots turned out to be real numbers and all solutions comprised of simple exponentials that either decayed to zero or blew up to infinity (or negative infinity) exponentially. Now we see that complex roots of the characteristic polynomial leads to solutions that have sine and/or cosine multiplied by an exponential. The sine and cosine produce oscillatory solutions (e.g. oscillations that you might expect from the mass-spring system of Figure 3.2).

Rather than simply write a solution symbolically, I think that it is important to be able to visualize what the function looks like graphically. Let's consider the function

$$y(t) = e^{-2t}\sin(10\,t). \tag{3.15}$$

This is Equation (3.13) with $c_3 = 0$ and $c_4 = 1$, so (3.15) is a solution to (3.8). In Figure 3.3 we show a plot of this solution (solid), along with its constituent parts (dashed).

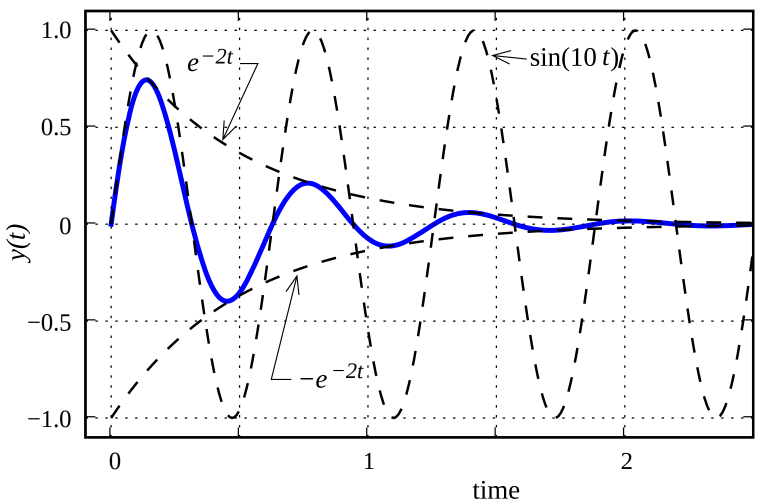

Figure 3.3: The solution (3.15) plotted as a solid curve, along with its parts (dashed).

Notice that one of the dashed curves in the figure is $\sin(10\,t)$; it oscillates between 1.0 and −1.0 at constant amplitude and period $T = 2\pi/10 = 0.6283$. The other dashed curve is e^{-2t}, a decaying exponential that loses 86% of its value every unit of time. Our solution (3.15) is what we get when we multiply these two functions together. The solution oscillates like the sine function (with the same frequency) except its amplitude decays exponentially, losing 86% of its value every unit of time. The solution cycles between e^{-2t} and $-e^{-2t}$. Some people say e^{-2t} is the "envelope" for the oscillatory solution (3.15).

Graphically, Figure 3.3 reminds us of the roles of the real and imaginary parts of the characteristic roots $s = -2 \pm i\,10$. Because the roots are complex, solutions are oscillatory. The imaginary part $\omega = 10$ is the "circular frequency" of the oscillation. The period is $2\pi/\omega$. The real part, $\sigma = -2$, is the exponential decay rate of the envelope of the oscillation.

Before moving on to the next section, you should take a moment to thing what the plot would look like if the characteristic roots were $s = +2 \pm 10\,i$? How about $s = 0 \pm 10\,i$?

3.3 Satisfying Initial Conditions

As before, the constants c_k in our general solutions (3.3, 3.13) to second order differential equations can be chosen to satisfy specific initial conditions. Here, I deliberately use the plural "condition**s**" because one condition (as we had in Chapter 2) is not enough for us now. To illustrate this, consider Figure 3.4 which shows several solutions to the differential equation (3.1).

Notice that all the solutions shown in the figure satisfy the initial condition $y(0) = 1.0$, yet they are distinct solutions. Notice that each of the solutions in the figure has a different slope

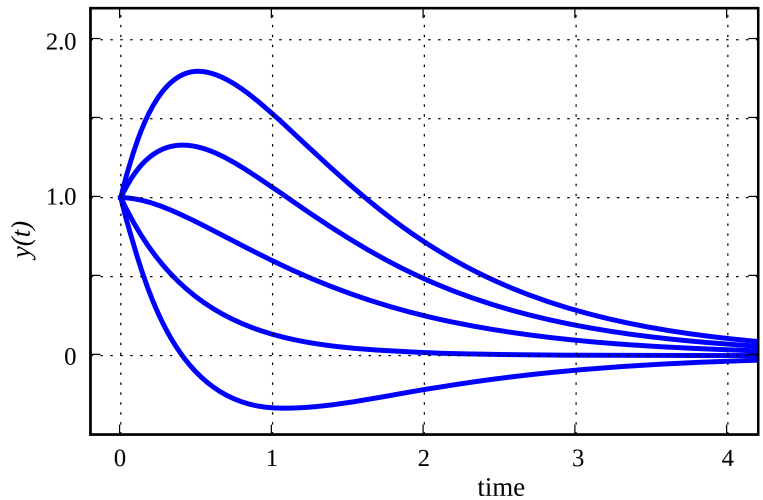

Figure 3.4: Several solutions to (3.1), each with with the same (partial) initial condition $y(0) = 1.0$. The slopes $\dot{y}(0)$, however, are all different at time $t = 0$. The slopes are 4.0, 2.0, 0.0, -2.0, and -4.0.

at $t = 0$. Therefore, it might seem reasonable when I tell you that if we specify the derivative $\dot{y}(0)$ at the initial time, along with the position $y(0)$, then the two arbitrary constants c_1 and c_2 in (3.3) can be determined uniquely. We get just one solution to the differential equation when the initial value and slope are specified.

This makes sense from a simple equation-counting perspective too (I think). Since we have two arbitrary constants, it is going to take two initial conditions (two equations) to specify the constants uniquely.

When we have 3^{rd}, 4^{th} and n^{th} order linear differential equations, the general solution will contain 3, 4, and n arbitrary constants, respectively. To specify a solution uniquely, one needs 3, 4, and n initial conditions (or boundary conditions) for these cases.

Before concluding this chapter, I wish to make one more observation regarding Figure 3.4. All five solutions depicted in the figure solve the differential equation (3.1). Hence, all five solutions have characteristic roots $s = -1$ and $s = -2$. Therefore, one might expect all five solutions to have the same relative decay rates toward zero.

When we look at the solutions in the figure, though, it appears that four of the solutions have similar decay rates. However, there is one solution (second from the bottom) which appears to go to zero faster than the others. Why is this?

It is a fluke of the initial condition. Recall that the general solution to (3.1) is

$$y(t) = c_1\, e^{-t} + c_2\, e^{-2t}.$$

The solution generally has a piece that decays like e^{-t} and another piece that decays a bit faster like e^{-2t}. The solution second from the bottom in Figure 3.4 has initial conditions $y(0) = 1$ and $\dot{y}(0) = -2.0$. To match these initial conditions, the coefficients in the general solution must be $c_1 = 0.0$ and $c_2 = 1.0$. Thus the e^{-t} is eliminated completely, just leaving the faster decaying e^{-2t}. This is a rare occurrence. Solutions normally have all pieces.

3.4 Take-Aways

The take-away concepts and ideas from Chapter 3 are outlined below.

1. Finding solutions to second order, constant coefficient, linear, homogeneous, differential equations
$$a\,\ddot{y}(t) + b\,\dot{y}(t) + c\,y(t) = 0,$$
is similar to finding solutions to the first order equations in Chapter 2: one seeks a solution of the form $y(t) = e^{st}$. Upon substitution into the differential equation, one obtains a second degree characteristic polynomial:
$$a\,s^2 + b\,s + c = 0,$$
whose roots are
$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2a}.$$

2. If the roots of the characteristic polynomial are real and $s_1 \neq s_2$, then the general solution of the differential equation takes the form
$$y(t) = c_1\,e^{s_1 t} + c_2\,e^{s_2 t}.$$
As discussed in Section 2.4, the signs of the roots s_1 and s_2 determine whether the solutions grow or decay, as well as the rate of growth or decay.

3. If the roots of the characteristic polynomial are complex, they come in a conjugate pair:
$$s_1 = \sigma + i\,\omega, \qquad s_2 = \sigma - i\,\omega.$$
In this case the general solution of the differential equation, takes the form
$$y(t) = c_1\,e^{\sigma t}\,\cos(\omega\,t) + c_2\,e^{\sigma t}\,\sin(\omega\,t).$$
The real part of the characteristic root, σ determines the rate of exponential growth or decay. The imaginary part of the characteristic root determines the circular frequency of the solution.

4. Regardless of whether the roots are real or complex, this type of differential equation *always* has two independent solutions, $y_1(t)$ and $y_2(t)$ which satisfy the differential equation individually. The *general solution* to the differential equation can be written as
$$y(t) = c_1\,y_1(t) + c_2\,y_2(t),$$
for arbitrary constants c_1 and c_2.

5. The arbitrary constants c_1 and c_2 can be used to satisfy initial conditions or boundary conditions. For example, if the values of $y(0)$ and $\dot{y}(0)$ are given, then one can determine values of c_1 and c_2 for the solution by solving the algebraic system of two equations with two unknowns:
$$\begin{aligned} c_1\,y_1(0) + c_2\,y_2(0) &= y(0), \\ c_1\,\dot{y}_1(0) + c_2\,\dot{y}_2(0) &= \dot{y}(0). \end{aligned}$$

Chapter 4

Higher Order, Constant Coefficient, Linear, Homogeneous Differential Equations

Now that we're experts at solving first and second order constant coefficient linear differential equations, solving higher order equations is fairly straightforward.

Let's discuss it in the context of an example:

$$\frac{d^5}{dt^5}y(t) + 9\,\frac{d^4}{dt^4}y(t) + 167\,\frac{d^3}{dt^3}y(t) + 803\,\frac{d^2}{dt^2}y(t) + 4916\,\frac{d}{dt}y(t) + 11544\,y(t) = 0. \tag{4.1}$$

Notice that it is a *fifth* order differential equation[1]. Nonetheless, we can tackle it in the same way. When we assume a solution of the form $y(t) = c\,e^{st}$, we end up with a characteristic equation in s that must be solved.

Although it's probably a good idea to do all the steps in detail a few more times (go ahead and do it!), there is a pattern that emerges that can save a lot of time. The characteristic polynomial associated with (4.1) is

$$s^5 + 9\,s^4 + 167\,s^3 + 803\,s^2 + 4916\,s + 11544 = 0. \tag{4.2}$$

For constant coefficient linear differential equations, the characteristic polynomial is always a polynomial whose degree is equal to the order of the differential equation. Furthermore, the coefficients of the polynomial are always the same as the coefficients to the differential equation.

4.1 Finding Characteristic Roots

Recall from the previous chapter that when we had a second order differential equation, it produced a quadratic characteristic equation similar to $as^2 + bs + c = 0$. In that case, we were able to easily find the roots according to the formula

$$s_1,\ s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{4.3}$$

[1] The choice of coefficients may seem a bit odd to you. However, they were selected for a specific purpose. You'll see later in the chapter.

Unfortunately, there is no *simple* formula for finding the roots of polynomial equations of degree 3 and higher. The best way to obtain the roots to higher order polynomials is to use a computer algorithm[2].

4.1.1 Polynomial Root Finding with Matlab

Throughout this semester, we will use Matlab extensively. As you probably know, Matlab is a powerful numerical analysis tool used throughout engineering and scientific practice. It is used heavily in control engineering too. Among many other things, Matlab is able to calculate our roots in a small fraction of a second.

It's easy to do. After firing up Matlab, type the following command at the prompt:

```
>> p = [1,9,167,803,4916,11544]
```

This is how we represent the polynomial on the left hand side of (4.2) within Matlab. This is also how one writes a 1×6 matrix in Matlab. Depending on the context in which it is used, Matlab knows how to interpret it.

Now, you just need to type

```
>> roots(p)
```

If everything worked correctly, Matlab should spit out the five roots: $s = -2 \pm i\,10$, $s = -1 \pm i\,6$, and $s = -3$.

We get five roots because (4.2) has a fifth order polynomial. Also, since all coefficients of the polynomial are real, we expected that any complex roots would come in conjugate pairs.

4.2 Constructing the General Solution

From the characteristic roots, it is a rather simple matter construct the general solution. The rule states that for each real root $s = \sigma$ of the characteristic polynomial, there is a solution to the original differential equation of the form $y(t) = e^{\sigma t}$. For each complex conjugate pair of roots $s = \sigma \pm i\,\omega$, there are two solutions of the form $y(t) = e^{\sigma t}\,\cos(\omega\,t)$ and $y(t) = e^{\sigma t}\,\sin(\omega\,t)$. Each of these solutions make up part of the general solution.

Returning back to our example in Equation (4.1), we find that general solution is

$$y(t) = c_1\,e^{-3t} + c_2\,e^{-t}\cos(6\,t) + c_3\,e^{-t}\sin(6\,t) + c_4\,e^{-2t}\cos(10\,t) + c_5\,e^{-2t}\sin(10\,t). \qquad (4.4)$$

Note that the general solution above has five (independent) components, with five arbitrary constants c_j. We get five components because there were five roots to the characteristic polynomial. In turn, the characteristic polynomial had five roots because the original differential equation was of fifth order.

In general, an n^{th} order, linear, homogeneous differential equation has a general solution with n independent components.

[2]Most computational algorithms find roots via a smart iterative trial and error process. If you've taken MEE 380/381, you should know all about this.

4.3 What about repeated roots?

Given the result described above, let's tackle another differential equation:

$$\frac{d^4}{dt^4}y(t) + 2\,\frac{d^3}{dt^3}y(t) + 46\,\frac{d^2}{dt^2}y(t) + 192\,\frac{d}{dt}y(t) + 200\,y(t) = 0. \tag{4.5}$$

In this case, the characteristic polynomial is

$$s^4 + 2\,s^3 + 46\,s^2 + 192\,s + 200 = 0, \tag{4.6}$$

which can be written as

$$(s^2 - 2\,s + 50)\,(s+2)^2 = 0.$$

Therefore, we have characteristic roots at $s = 1 \pm 7\,i$ as well as repeated roots at $s = -2, -2$.

Now, we might be tempted to write the general solution to (4.5) as

$$y(t) = c_1\,e^t\,\cos(7\,t) + c_2\,e^t\,\sin(7\,t) + c_3\,e^{-2t} + c_4\,e^{-2t}.$$

But this clearly is **not correct**. The last two terms can be combined into one term: $c_3\,e^{-2t} + c_4\,e^{-2t} \to c_5\,e^{-2t}$. This does not have four independent terms as required since (4.5) is a fourth order differential equation. We're missing a term.

It turns out that the missing term looks like $t\,e^{-2t}$, and the **correct** general solution is

$$y(t) = c_1\,e^t\,\cos(7\,t) + c_2\,e^t\,\sin(7\,t) + c_3\,e^{-2t} + c_4\,t\,e^{-2t}. \tag{4.7}$$

Now you might wonder where that new term came from. And if you are wondering, I urge you to go look it up in a real textbook. For purposes of this course, I'd argue that we really don't care about the case of repeated roots. Watch what happens if we change one of the coefficients in (4.5) from 192 to 192.01. In this case, the characteristic roots become $s = 1.0001 \pm 7.0001\,i$, $s = -1.9815$, and $s = -2.0186$. I could change any of the coefficients of (4.5) and the roots would no longer be repeated. In this sense, having repeated roots is rare. If we were to pick a constant coefficient linear differential equations at random, the probability of picking one with repeated roots would be zero.

There is one exception. Later, we'll find that is somewhat common to have repeated roots at $s = 0$. We'll tackle these later.

4.4 Take-Aways

The take-away concepts and ideas from Chapter 4 are outlined below.

1. One can extend ideas for solving first and second order differential equations in Chapters 2 and 3 to solving higher order constant coefficient, linear, homogeneous differential equations of the form:

$$a_n\,\frac{d^n}{dt^n}y(t) + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\,\frac{d}{dt}y(t) + a_0\,y(t) = 0. \tag{4.8}$$

   In seeking a solution of the form $y(t) = e^{st}$, we get a characteristic equation of the form.

$$a_n\,s^n + a_{n-1}\,s^{n-1} + \ldots + a_1\,s + a_0 = 0 \tag{4.9}$$

2. The *general solution* to the differential equation (4.8) contains n independent components and can be written as

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + c_3\, y_3(t) + \ldots + c_n\, y_n(t).$$

   The n components are determined by the n roots of the characteristic polynomial (4.9).

3. When the degree of the characteristic polynomial (4.9) is bigger than two ($n > 2$), the most practical way to find characteristic roots is usually to use some computer program. Section 4.1.1 shows you how to do it with Matlab.

4. For each real characteristic root $s = \sigma_j$, the general solution has a component of the form

$$y_j(t) = e^{\sigma_j t}.$$

5. For each complex conjugate pair of characteristic roots $s = \sigma_j + i\,\omega_j$, $s = \sigma_j - i\,\omega_j$, the general solution has components of the form

$$y_j(t) = e^{\sigma_j t}\,\cos(\omega_j\, t) \qquad y_{j+1}(t) = e^{\sigma_j t}\,\sin(\omega_j\, t).$$

6. If some of the characteristic roots are repeated, the corresponding solution components take a somewhat different form. However we will not be concerned with these since they are unlikely to occur in normal circumstances.

7. For an n^{th} order differential equation, one needs n initial conditions to uniquely determine the constants c_1 through c_n.