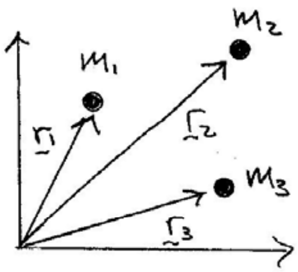
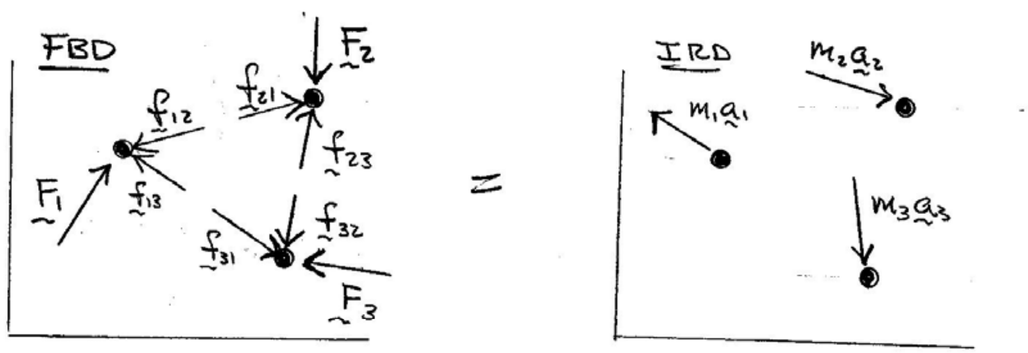
Dynamics of a Collection of Particles



Suppose we have 3 particles moving in a two-dimensional space

The FBD & IRD are shown below:



Here $\underset{\sim}{f}_{12}$ is the force acting on particle 1 due to particle 2

For example if particles 1 & 2 are connected by a spring, then $\underset{\sim}{f}_{12}$ is the spring force on particle 1 due to this spring. $\underset{\sim}{f}_{21}$ is the spring force on particle 2

In general $\underset{\sim}{f}_{ji}$ is the force on particle j due to particle i

We say that the $\underset{\sim}{f}_{ji}$ are <u>internal forces</u>. They are forces coming from within the system of particles.

The $\underset{\sim}{F}_j$ are <u>external forces</u>, forces coming from outside the system of particles

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Applying Newton's Second Law to each of the particles, we obtain

Particle 1: $\underset{\sim}{F}_1 + \underset{\sim}{f}_{12} + \underset{\sim}{f}_{13} = m_1 \underset{\sim}{a}_1$

Particle 2: $\underset{\sim}{F}_2 + \underset{\sim}{f}_{21} + \underset{\sim}{f}_{23} = m_2 \underset{\sim}{a}_2$

Particle 3: $\underset{\sim}{F}_3 + \underset{\sim}{f}_{31} + \underset{\sim}{f}_{32} = m_3 \underset{\sim}{a}_3$

If we add all three pieces together, we get

$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 + \left(\underset{\sim}{f}_{12} + \underset{\sim}{f}_{21}\right) + \left(\underset{\sim}{f}_{13} + \underset{\sim}{f}_{31}\right) + \left(\underset{\sim}{f}_{23} + \underset{\sim}{f}_{32}\right)$$

$$= m_1 \underset{\sim}{a}_1 + m_2 \underset{\sim}{a}_2 + m_3 \underset{\sim}{a}_3$$

Now...

The forces between particles must abide by Newton's 3rd Law

$$\Rightarrow \mathbf{f}_{12} = -\mathbf{f}_{21} \;;\; \mathbf{f}_{13} = -\mathbf{f}_{31} \;;\; \mathbf{f}_{23} = -\mathbf{f}_{31}$$

$\Rightarrow$ All the terms in parentheses in (*) must be zero. All internal forces vanish

$$\Rightarrow \underbrace{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3}_{\text{This is the total external force acting on the system}} = \underbrace{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m\,\mathbf{a}_3}_{\text{But what is this?}} \quad (\#)$$

Recall definition of center of mass

$$m_T\,\mathbf{r}_G = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$$

where $m_T = m_1 + m_2 + m_3$

Let's take a time derivative

$$\Rightarrow m_T\,\dot{\mathbf{r}}_G = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + m_3\dot{\mathbf{r}}_3$$

velocities

or $m_T\mathbf{V}_G = m_1\mathbf{V}_1 + m_2\mathbf{V}_2 + m_3\mathbf{V}_3$

Taking the second time derivative, we get

$$m_T \ddot{\underset{\sim}{r}}_G = m_1 \ddot{\underset{\sim}{r}}_1 + m_2 \ddot{\underset{\sim}{r}}_2 + m_3 \ddot{\underset{\sim}{r}}_3$$

or

$$m_T \underset{\sim}{a}_G = m_1 \underset{\sim}{a}_1 + m_2 \underset{\sim}{a}_2 + m_3 \underset{\sim}{a}_3$$

Therefore, we can re-write (#) as

$$\boxed{F_1 + F_2 + F_3 = m_T a_G}$$

The sum of the <u>EXTERNAL</u> forces is equal to the total mass times the acceleration of the center of mass

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This result extends to any number of particles an also works in 3 dimensions

$$\boxed{\sum_{j=1}^{N} \underset{\sim}{F}_j = m_T \underset{\sim}{a}_G}$$

Ta-daa!