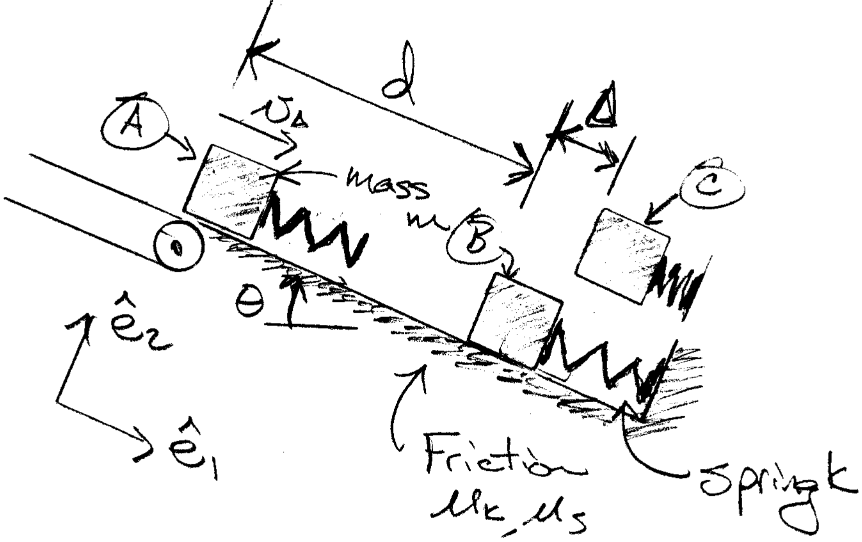
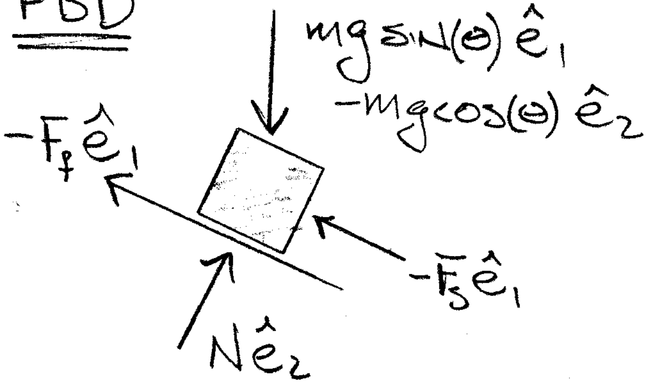


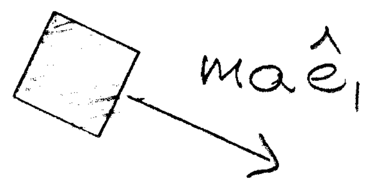
Given: $m, \mu_k, \theta, k,$ d, F_{max}

Find: Maximum speed v_A for which impact force does not exceed F_{max}

FBD



MAD



Newton $\hat{e}_2$: $N - mg\cos(\theta) = 0$ ← No accel in $\hat{e}_2$

$$\Rightarrow N = mg\cos(\theta)$$

So, while block is sliding to the right, friction is to the left

$$F_f = \mu_k N = \mu_k mg\cos(\theta)$$

Also, the spring force: $F_s = k\delta$ ← Amount the spring is compressed

Note: The "impact" force is that of the spring. The spring force is largest at maximum compression

$$\Rightarrow F_{max} = K\Delta$$ ← max spring compression

max spring force

$$\Rightarrow \Delta = \frac{F_{max}}{K} \quad (1)$$

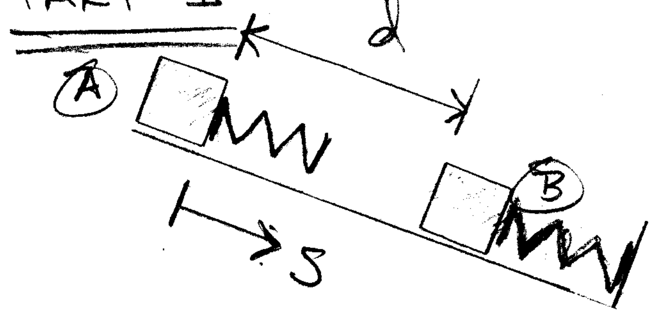
- Let state A be when block is at top of ramp.
- Let state B be when spring first touches the wall
- Let state C be when spring is at maximum compression

We'll split the problem into two parts:

PART I : From A to B

PART II : From B to C

PART I



Time: t

$t=0$ at A; $t=t_B$ at B

Position: $s\hat{e}_1$

$s=0$ at A; $s=d$ at B

Velocity: $\dot{s}\hat{e}_1$

$\dot{s}_1 = v_A$ at A; $\dot{s}_1 = v_B$ at B

Newton: $\hat{e}_1$: $mg\sin(\theta) - mg\mu_k\cos(\theta) = m\ddot{s}$

$\Rightarrow$ acceleration $\ddot{s} = g[\sin(\theta) - \mu_k\cos(\theta)]$

Velocity:

$$\dot{s} = \int g[\sin(\theta) - \mu_k\cos(\theta)]\,dt$$

$$= g[\sin(\theta) - \mu_k\cos(\theta)]\,t + C_0$$

I.C. $\dot{s}(0) = v_A$

$$\Rightarrow g[\quad]\cdot 0 + C_0 = v_A \Rightarrow C_0 = v_A$$

$$\Rightarrow \dot{s}(t) = g[\sin(\theta) - \mu_k\cos(\theta)]\,t + v_A \qquad (2)$$

Position

$$s(t) = \int \dot{s}(t)\,dt = \frac{1}{2}g[\sin(\theta) - \mu_k\cos(\theta)]\,t^2 + v_A t + C_1$$

I.C. $s(0) = 0$

$$\Rightarrow \frac{1}{2}g[\quad]\cdot 0^2 + v_A\cdot 0 + C_1 = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow s(t) = \frac{1}{2}g[\sin(\theta) - \mu_k\cos(\theta)]\,t^2 + v_A t$$

B.C. $s(t_B) = d$

$$\Rightarrow \frac{1}{2} g\left[\sin(\theta) - \mu_k \cos(\theta)\right] t_B^2 + v_A t_B = d$$

or $$\frac{1}{2} g\left[\sin(\theta) - \mu_k \cos(\theta)\right] t_B^2 + v_A t_B - d = 0$$

Use quadratic formula to solve for t_B

$$\Rightarrow t_B = \frac{-v_A \pm \sqrt{v_A^2 + 2g\left[\sin(\theta) - \mu_k \cos(\theta)\right] d}}{g\left[\sin(\theta) - \mu_k \cos(\theta)\right]} \quad (3)$$

Observe that this provides two roots

→ if $\sin(\theta) - \mu_k \cos(\theta) > 0$, then you should see that one root is positive while the other is negative. Since state B happens AFTER state A, need $t_B > 0 \Rightarrow$ "+" root

$$t_B = \frac{-v_A + \sqrt{v_A^2 + 2gd\left[\sin(\theta) - \mu_k \cos(\theta)\right]}}{g\left[\sin(\theta) - \mu_k \cos(\theta)\right]} \quad (4)$$

→ if $\sin(\theta) - \mu_k \cos(\theta) < 0$, then it's possible that the term under $\sqrt{\ }$ is negative. This would mean that the spring never touches the wall.

However, we are supposed to find maximum v_A for which the spring force isn't too big. So we'll always choose a large enough v_A so that

$$v^2 + 2g\left[\sin(\theta) - \mu_k \cos(\theta)\right] > 0$$

In this case, we get one positive & one negative root for (3) again, and we choose "+" to retain the positive root.

Therefore (4) works for the case $\sin(\theta) - \mu_k \cos(\theta) < 0$ as well.

Now substitute (4) into (2) to find velocity at state B.

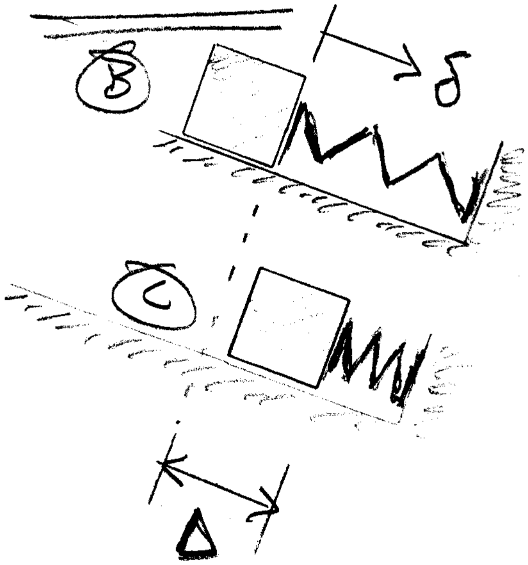
$$\dot{s}(t_B) = v_B$$

$$\Rightarrow g\left[\sin(\theta) - \mu_k \cos(\theta)\right] \left[\frac{-v_A + \sqrt{v_A^2 + 2gd\left[\sin(\theta) - \mu_k \cos(\theta)\right]}}{g\left[\sin(\theta) - \mu_k \cos(\theta)\right]}\right] + v_A = v_B$$

So

$$v_B = \sqrt{v_A^2 + 2gd\left[\sin(\theta) - \mu_k \cos(\theta)\right]} \quad (5)$$

PART II



Time : τ

$\tau = 0$ at B ; $\tau = \tau_c$ at C

Position: $\delta \hat{e}_1$

$\delta = 0$ at B ; $\delta = \Delta$ at C

Velocity: $\dot{\delta} \hat{e}_1$

$\dot{\delta} = v_B$ at B ; $\dot{\delta} = 0$ at C

Newton: $\hat{e}_1$: $mg\sin(\theta) - mg\mu_k\cos(\theta) - K\delta = m\ddot{\delta}$

or $$m\ddot{\delta} + K\delta = mg[\sin(\theta) - \mu_k\cos(\theta)] \quad (6)$$

Note: This is a 2^{nd} order, inhomogeneous, ODE
non homogeneous?

The general sol'n to (6) has a homogeneous part $\delta_h(t)$ which is the sol'n to

$$m\ddot{\delta} + K\delta = 0$$

Seek soln of form $\delta = e^{\lambda\tau} \Rightarrow \ddot{\delta} = \lambda^2 e^{\lambda\tau}$

Subs $\Rightarrow m\lambda^2 e^{\lambda\tau} + K e^{\lambda\tau} = 0$

$\Rightarrow (m\lambda^2 + K)e^{\lambda\tau} = 0$

$\Rightarrow m\lambda^2 + K = 0 \Leftarrow$ characteristic eq'n

$\Rightarrow \lambda = \pm i\sqrt{\frac{K}{m}}$

So homogeneous sol'n

$$\delta_h(t) = P\cos(\omega\tau) + Q\sin(\omega\tau)$$

where $\omega = \sqrt{\frac{K}{m}}$ (6½) and P & Q are constants to be determined

Now the particular Sol'n. Since the RHS of (6) is constant the particular sol'n is also constant

$$\delta = \delta_p = \text{const} \ ; \ \ddot{\delta} = \dot{\delta} = 0$$

Subst into (6)

$$m \cdot 0 + K\delta_p = mg[\sin(\theta) - \mu_k\cos(\theta)]$$

$$\Rightarrow \delta_p = \frac{mg}{K}[\sin(\theta) - \mu_k\cos(\theta)]$$

General sol'n to (6) is

$$\delta(\tau) = P\cos(\omega\tau) + Q\sin(\omega\tau) + \frac{mg}{K}[\sin(\theta) - \mu_k\cos(\theta)] \quad (7)$$

I.C.: $\delta(0) = 0$

$$\Rightarrow P\underbrace{\cos(0)}_{1} + Q\underbrace{\sin(0)}_{0} + \frac{mg}{K}[\sin(\theta) - \mu_k\cos(\theta)] = 0$$

$$\Rightarrow P + \frac{mg}{K}[\sin(\theta) - \mu_k\cos(\theta)] = 0$$

$$\Rightarrow P = -\frac{mg}{K}[\sin(\theta) - \mu_k\cos(\theta)] \quad (8)$$

I.C. $\dot{\delta}(0) = v_B$

$$\Rightarrow -\omega P \overset{0}{\sin(0)} + \omega Q \overbrace{\cos(0)}^{1} = v_B$$

$$\Rightarrow \omega Q = v_B \Rightarrow Q = \frac{v_B}{\omega} \quad (8\tfrac{1}{2})$$

Now let's look at boundary conditions at C

B.C. $\delta(\tau_c) = \Delta$

$$\Rightarrow P\cos(\omega\tau_c) + Q\sin(\omega\tau_c) + \overbrace{\frac{mg}{k}\left[\sin(\theta) - \mu_k\cos(\theta)\right]}^{\text{This is } -P} = \Delta$$

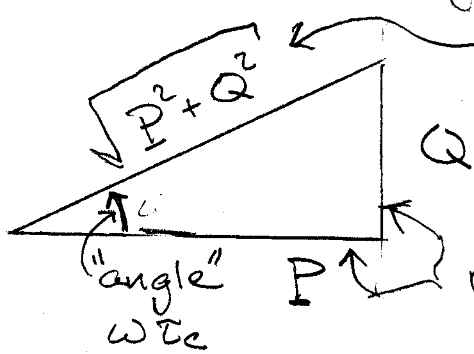
$$\Rightarrow P\cos(\omega\tau_c) + Q\sin(\omega\tau_c) = P + \Delta \quad (9)$$

B.C. $\dot{\delta}(\tau_c) = 0$

$$\Rightarrow -\omega P\sin(\omega\tau_c) + \omega Q\cos(\omega\tau_c) = 0$$

$$\Rightarrow \frac{\sin(\omega\tau_c)}{\cos(\omega\tau_c)} = \frac{Q}{P} \Rightarrow \tan(\omega\tau_c) = \frac{Q}{P} \quad (10)$$

Now, we can interpret the result above geometrically



obtain hypotenuse via Pythagorean theorem

ratio of these two sides is related to the "angle" through (10)

So $$\cos(\omega\tau_c) = \frac{P}{\sqrt{P^2+Q^2}} \quad (11) \quad ; \quad \sin(\omega\tau_c) = \frac{Q}{\sqrt{P^2+Q^2}} \quad (12)$$

Now substitute (11) & (12) into (9)

$$\frac{P^2}{\sqrt{P^2+Q^2}} + \frac{Q^2}{\sqrt{P^2+Q^2}} = P+\Delta$$

$$\Rightarrow \frac{P^2+Q^2}{\sqrt{P^2+Q^2}} = P+\Delta \Rightarrow \sqrt{P^2+Q^2} = P+\Delta$$

$$\Rightarrow \cancel{P^2}+Q^2 = \cancel{P^2}+2P\Delta+\Delta^2$$

Substitute (6½), (8), & (8½) into expression above, we get

$$\frac{m v_B^2}{k} = -\frac{2mg}{k}\left[\sin(\theta)-\mu_k\cos(\theta)\right]\Delta + \Delta^2$$

$$\Rightarrow v_B^2 = \frac{k}{m}\Delta^2 - 2g\left[\sin(\theta)-\mu_k\cos(\theta)\right]\Delta$$

Substitute (5) into expression above

$$v_A^2 + 2g\left[\sin(\theta)-\mu_k\cos(\theta)\right]d = \frac{k}{m}\Delta^2 - 2g\left[\sin(\theta)-\mu_k\cos(\theta)\right]\Delta$$

So

$$v_A = \sqrt{\frac{k}{m}\Delta^2 - 2g\left[\sin(\theta)-\mu_k\cos(\theta)\right](d+\Delta)}$$

Substitute (1) into expression above

$$\boxed{v_A = \sqrt{\frac{F_{max}}{mK} - 2g\left[\sin(\theta) - \mu_K \cos(\theta)\right]\left(d + \frac{F_{max}}{K}\right)}}$$

This is exactly the same solution we found with the Work-Energy method.

Units were verified previously.